

Momentenmethode

R.F. Harrington: Field Computation by Moment Methods.
McMillan, New York

linearer Operator $\mathcal{L}\{\Phi\} = g$

Basisfunktionen $\Phi(\vec{r}) \approx \Phi_N(\vec{r}) = \sum a_j b_j(\vec{r})$

Skalarprodukt $s = \langle q, p \rangle$

Testfunktionen $t_i(\vec{r}) \rightarrow q_i = \langle q, t_i \rangle$

$$\mathcal{L}\{\Phi(\vec{r})\} = \mathcal{L}\left\{\sum a_j b_j(\vec{r}) + \delta\Phi(\vec{r})\right\} = \sum a_j \mathcal{L}\{b_j(\vec{r})\} + \mathcal{L}\{\delta\Phi(\vec{r})\}$$

$$\langle \mathcal{L}\{\Phi(\vec{r})\}, t_i \rangle = \sum a_j \langle \mathcal{L}\{b_j(\vec{r})\}, t_i \rangle + \langle \mathcal{L}\{\delta\Phi(\vec{r})\}, t_i \rangle = \langle g, t_i \rangle$$

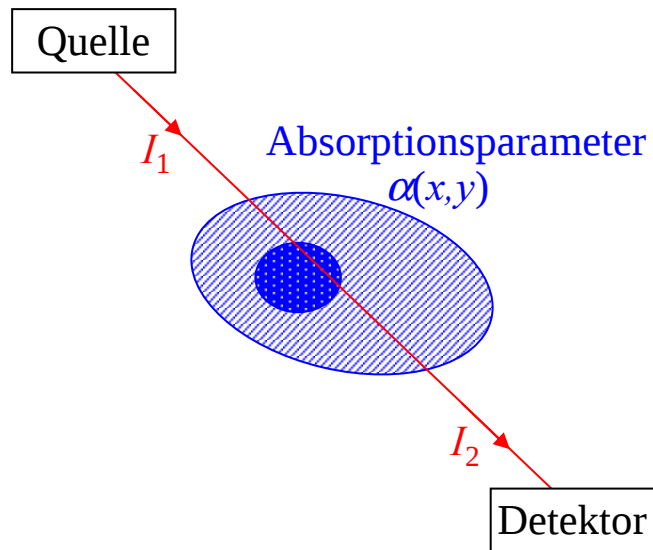
$$L_{i,j} = \langle \mathcal{L}\{b_j(\vec{r})\}, t_i \rangle$$

$$g_i = \langle g, t_i \rangle$$

$$\delta_i = \langle \mathcal{L}\{\delta\Phi(\vec{r})\}, t_i \rangle$$

$$[L_{i,j}] [a_j] = [g_i] - [\delta_i]$$

Beispiel Computer Tomographie



Strahl $\vec{r}_{a\varphi}(s) = s \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix} + a \begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix}$

$$\frac{d}{ds} I = -\alpha(\vec{r}_{a\varphi}(s)) I$$

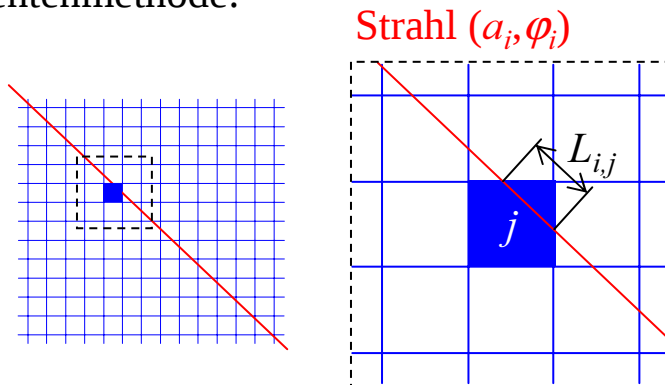
$$-\frac{dI}{I} = \alpha(\vec{r}_{a\varphi}(s)) ds$$

$$M(a, \varphi) = \ln(I_1/I_2) = \int \alpha(\vec{r}_{a\varphi}(s)) ds$$

Meßgröße

Unbekannte

Momentenmethode:



Basisfunktionen:

$$b_j(x, y) = \begin{cases} 1 & \text{falls } \vec{r} \in j \\ 0 & \text{sonst} \end{cases}$$

Testmethode: „Pointmatching“
(echte Meßpunkte)

$$g_i = M(a_i, \varphi_i)$$

$$\rightarrow [L_{i,j}][a_j] \approx [g_i]$$

Beispiel: Momentenmethode

$$f''(x) = y(x)$$

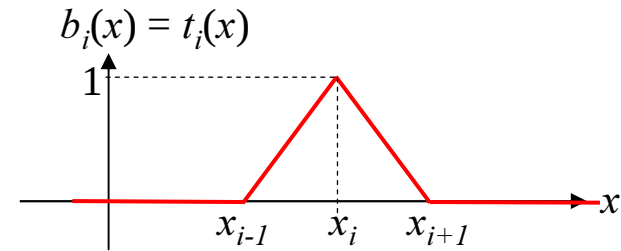
$$f(a) = \alpha$$

$$f(b) = \beta$$

$$i = 0, 1, \dots, I$$

$$\Delta x = (b - a) / I$$

$$x_i = a + i\Delta x$$



$$f_0 = \alpha$$

$$f_I = \beta$$

$$g_i = \int y(x) t_i(x) dx$$

$$\frac{1}{\Delta x} \begin{bmatrix} -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ \vdots \\ f_{I-3} \\ f_{I-2} \\ f_{I-1} \end{bmatrix} = \begin{bmatrix} g_1 - f_0 / \Delta x \\ g_2 \\ g_3 \\ \vdots \\ \vdots \\ g_{I-3} \\ g_{I-2} \\ g_{I-1} - f_I / \Delta x \end{bmatrix}$$

Beispiel ...

$$f''(x) = y(x)$$

$$f(a) = \alpha$$

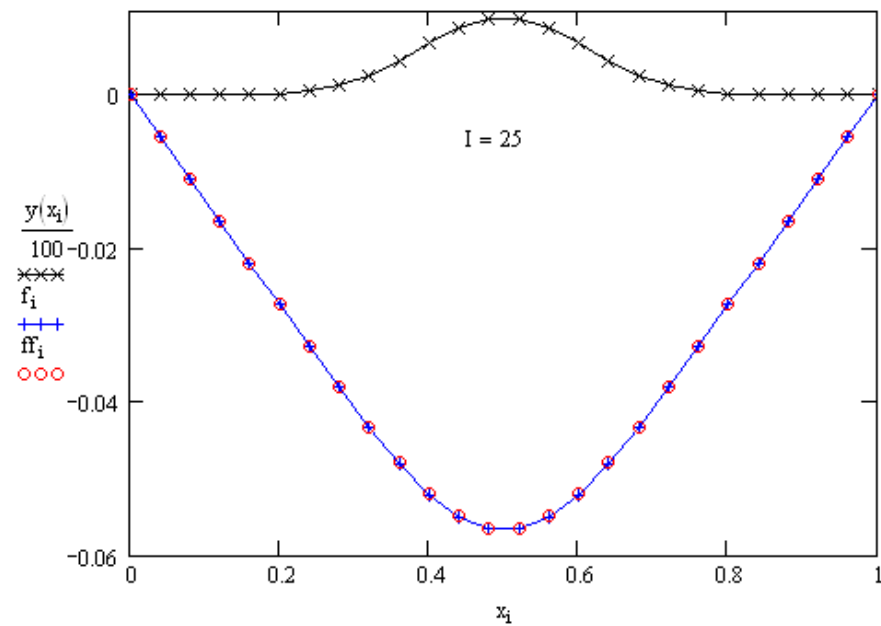
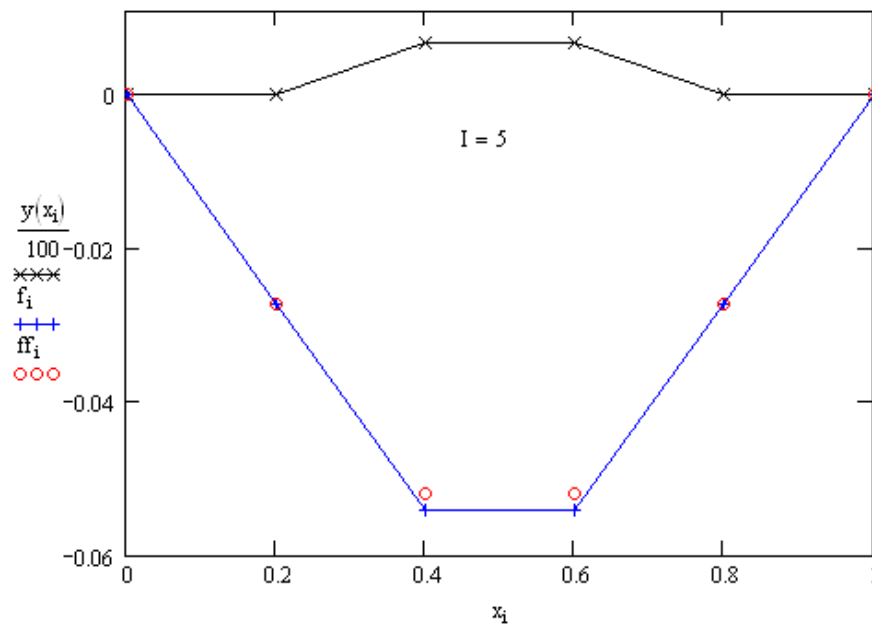
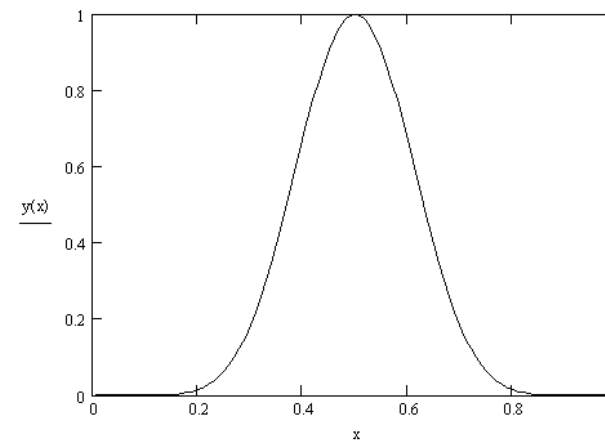
$$f(b) = \beta$$

$$a := 0 \quad b := 1 \quad I := 5 \quad i := 0..I \quad \Delta x := \frac{b-a}{I} \quad x_i := a + i \cdot \Delta x$$

$$f_0 := 0 \quad f_I := 0 \quad y(x) := \sin(x \cdot \pi)^8$$

FD-Methode

Momenten Methode



Minimale Energie des Fehlerfeldes

$$\nabla^2 \Phi = g = -\frac{\rho}{\varepsilon_0}$$

$$\operatorname{div}(\vec{E}_N + \delta \vec{E}) = g = -\frac{\rho}{\varepsilon_0}$$

$$\vec{E}_N = -\nabla \Phi_N = -\sum a_i \nabla b_i$$

$$\delta \vec{E} = -\nabla \delta \Phi$$

$$\int \|\vec{E}\|^2 dV = \int \left(\|\vec{E}_N\|^2 + 2\vec{E}_N \cdot \delta \vec{E} + \|\delta \vec{E}\|^2 \right) dV$$

$$\int \|\delta \vec{E}\|^2 dV = \int \|\vec{E}\|^2 dV + \int \left(\|\vec{E}_N\|^2 - 2\vec{E}_N \cdot \vec{E} \right) dV$$

1. Greenscher Satz

$$\int \|\delta \vec{E}\|^2 dV = \text{const} + \int \left(\|\nabla \Phi_N\|^2 + 2\Phi_N g \right) dV - \underbrace{\int_{\partial V} \Phi_N \nabla \Phi \cdot d\vec{A}}_0$$

$$2I(\underline{a}) = \int \left(\|\nabla \Phi_N\|^2 + 2\Phi_N g \right) dV$$

für homogene Dirichlet
oder Neumann RB

Variationsmethode $\leftrightarrow$ Galerkin-Methode

$$\mathcal{L}\{\Phi\} = \nabla^2 \Phi = g \quad \leftrightarrow \quad I(a) = \frac{1}{2} \int (\|\nabla \Phi_N\|^2 + 2g\Phi_N) dV$$

$$2I(\mathbf{a}) = \sum a_i a_j \underbrace{\int \nabla b_i \cdot \nabla b_j dV}_{-\tilde{L}_{i,j}} + 2 \sum a_i \underbrace{\int b_i g dV}_{g_i}$$

$$2I(\mathbf{a}) = -\mathbf{a}^t \tilde{\mathbf{L}} \mathbf{a} + 2\mathbf{a}^t \mathbf{g} \stackrel{!}{=} \min \quad \leftrightarrow \quad \tilde{\mathbf{L}} \mathbf{a} = \mathbf{g}$$

1. Greenscher Satz

$$L_{i,j} = \langle \mathcal{L}\{b_j(\vec{r})\}, b_i \rangle = \int (\nabla^2 b_j) b_i dV \stackrel{\downarrow}{=} - \underbrace{\int \nabla b_i \cdot \nabla b_j dV}_{\tilde{L}_{i,j}} + \int_{\partial V} b_i \nabla b_j \cdot d\vec{A}$$

$$g_i = \langle b_i, g \rangle$$

$$\mathbf{L} \mathbf{a} = \mathbf{g}$$

$$\begin{aligned}\operatorname{div} \vec{D} &= \rho \\ \operatorname{rot} \vec{E} &= -\frac{\partial}{\partial t} \vec{B} \\ \operatorname{div} \vec{B} &= 0 \\ \operatorname{rot} \vec{H} &= \vec{J} + \frac{\partial}{\partial t} \vec{D}\end{aligned}$$

$$\begin{aligned}\vec{D} &= \varepsilon \vec{E} \\ \vec{B} &= \mu \vec{H} \\ \vec{J} &= \kappa \vec{E} + \vec{J}_q\end{aligned}$$

RB

$$\frac{\partial}{\partial t} \rightarrow j\omega$$

„div“ Gleichungen
implizit erfüllt

$$\operatorname{rot} \left(\frac{1}{\mu} \operatorname{rot} \vec{E} \right) - (\omega^2 \varepsilon - j\omega\kappa) \vec{E} = -j\omega \vec{J}_q$$

$$\vec{H} = \frac{1}{-j\omega\mu} \operatorname{rot} \vec{E}$$

$$\begin{aligned}\vec{D} &= \varepsilon \vec{E} \\ \vec{B} &= \mu \vec{H} \\ \vec{J} &= \kappa \vec{E} + \vec{J}_q\end{aligned}$$

RB

Diskretisierung
Kontinuisierung

$$\mathbf{L}\mathbf{e} = \mathbf{j}_q$$

$$\begin{aligned}\tilde{\mathbf{S}}\mathbf{d} &= \mathbf{q} \\ \mathbf{C}\mathbf{e} &= -\frac{\partial}{\partial t} \mathbf{b} \\ \mathbf{S}\mathbf{b} &= \mathbf{0} \\ \tilde{\mathbf{C}}\mathbf{h} &= \mathbf{j} + \frac{\partial}{\partial t} \mathbf{d}\end{aligned}$$

$$\begin{aligned}\mathbf{d} &= \mathbf{D}_\varepsilon \mathbf{e} \\ \mathbf{D}_{\mu^{-1}} \mathbf{b} &= \mathbf{h} \\ \mathbf{j} &= \mathbf{D}_\kappa \mathbf{e} + \mathbf{j}_q\end{aligned}$$

$$\tilde{\mathbf{C}} \mathbf{D}_{\mu^{-1}} \mathbf{C} \mathbf{e} - (\omega^2 \mathbf{D}_\varepsilon - j\omega \mathbf{D}_\kappa) \mathbf{e} = -j\omega \mathbf{j}_q$$

$$\mathbf{h} = \frac{1}{-j\omega} \mathbf{D}_{\mu^{-1}} \mathbf{C} \mathbf{e}$$

$$\begin{aligned}\mathbf{d} &= \mathbf{D}_\varepsilon \mathbf{e} \\ \mathbf{D}_{\mu^{-1}} \mathbf{b} &= \mathbf{h} \\ \mathbf{j} &= \mathbf{D}_\kappa \mathbf{e} + \mathbf{j}_q\end{aligned}$$

$$\mathbf{L}\mathbf{e} = \mathbf{j}_q$$